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FITTING A HYPERBOLIC SINE CURVE
TO CREEP DATA

by Max Shiffman & Bernard Friedman

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A Note Submitted
by the
Applied Mathematics Group, New York University
to the
Applied Mathematics Panel
National Defense Research Committee

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This note was prepared as a result of a suggestion from Mr. Ward F. Davidson, Consultant to the Chairman, NDRC, that a more exact method could be developed for adjusting a curve to creep data than those usually employed or the one suggested by McVetty in a recent issue of the Transactions of the American Society of Mechanical Engineers. It is hoped that the method outlined here will be useful to investigators and engineers concerned with the problems of materials under creep conditions.

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FITTING A HYPERBOLIC SINE CURVE TO CREEP DATA.

By M. Shiffman and B. Friedman

Various more or less empirical formulas have been used to connect the rate of creep of a metal (at a constant temperature) with the stress applied. The power function $r = a\,s^n$ where r is the rate of creep, s is the applied stress, and a, n are constants, has frequently been tried because of its simplicity. However, this formula often leads to very incorrect conclusions, especially when extrapolating from experimentally determined points to smaller stresses. It is more appropriate to use a relation which has some theoretical justification, provided that this can be done without too much labor and, on the other hand, gives a better fit. The "hyperbolic sine law"

(1) $$r = a \sinh bs$$

has been suggested on theoretical grounds by Prandtl,[1] Nadai[2] and Eyring,[3] and the question is to fit this formula to a set of experimentally determined points by finding suitable a and b.

This problem is discussed in a paper by McVetty in the Transactions of the American Society of Mechanical Engineers, October 1943. There a graphical method is exhibited for determining the constants a, b in the case of two given points. When there are more than two given points, McVetty determines a, b for all pairs of points and selects the average of these as the appropriate a, b. This procedure turns out to be reasonably accurate only if the points happen to fit an exponential curve very closely. While this is the case in McVetty's

example, it need not always be true. See illustrative data on page 4 , taken from the Creep Rate Book.

In this memorandum other possible procedures for determining a, b are developed which are valid in any case where the hyperbolic sine law is applicable, and are easily handled numerically.

1. Large Stresses. In the first place, if s is large (say $bs > 2$) the equation (1) can be approximated by

$$r = \frac{a}{2} e^{bs} ,$$

or (2)

$$\log r = \log \frac{a}{2} + b \cdot \log e \cdot s.$$

This is a straight line on semi-logarithm paper, and the constants a, b can be determined by standard methods. While the assumption of large s is not always correct, this happens to be the case in McVetty's illustration, and in fig. 1 we have plotted the data on semi-log paper. The data fits a straight line closely; measuring the slope and intercept yields the value of a and b. The results are $b = \frac{1}{624}$, $a = .0021$. Thus, we have quickly obtained

$$r = .0021 \sinh \frac{s}{624}$$

which fits the experimental points particularly closely. Even in this case I, the procedure here suggested might have advantages over McVetty's.

II. Arbitrary Stresses. In the general case the experimental data plotted on semi-logarithm paper will not fit a straight line closely but will descend more rapidly for smaller values of s. Compare with the hyperbolic sine curve given in

fig. 2. A least square criterion can be used to determine the best a, b.

Replace $r = a \sinh bs$

by

(3) $\log r = A + \log \sinh bs$, where $A = \log a$,

and apply the least squares criterion to (3), that is, determine A, b so as to minimize

(4) $$\sum_{i=1}^{n} (\log r_i - A - \log \sinh bs_i)^2 \quad .$$

This seems necessary since the stresses s_i are determined accurately while the experimental error of the creep rates r_i are undoubtedly of the percentage type, rather than the absolute type. This justifies the use of $\log r_i$ in place of r_i.

The equations obtained by minimizing (4) contain A and b in a transcendental manner. But this set of equations can easily be solved by a method of iteration, of which the first step will usually suffice.

The procedure to follow in making the approximation will be described; an example will be worked out; and then the derivation of the equations used will be given at the end:

Procedure:

First, plot the experimental points on a semi-logarithm paper and determine graphically a straight line which approximately fits the points. The slope of this straight line will yield a first approximation b_o to the value of b. The method is the same as in part I.

The first approximation A_o to the value of A is obtained from

$$(5) \qquad A_o = \frac{1}{n} \sum_{i=1}^{n} (\log r_i - \log \sinh b_o s_i).$$

The corrections Δb, ΔA to be added to these values b_o, A_o are found as follows:

Define the "weights" w_i by

$$w_i = s_i \coth b_o s_i.$$

Let $\bar{w}$ be the average of these weights, $\bar{w} = \frac{1}{n} \sum w_i$. Then

$$(6) \quad \Delta b = \frac{\sum (w_i - \bar{w}) (\log r_i - \log \sinh b_o s_i)}{.4343 \times \sum (w_i - \bar{w})^2}$$

and

$$(7) \qquad \Delta A = -.4343\, \bar{w} \cdot \Delta b.$$

The quantity .4343 is log e.

Example:

Consider the following data taken from the Creep Data Handbook, 1939, page 245, for a wrought iron bar at 1200° F under stress for 1,000 hours.

s(lb./in.^{2})	r(% per 10,000 hrs.)
400	.39
750	.90
1250	1.59
1750	3.32
2500	8.75

Plotting these points on semi-log paper (fig. 2) and selecting a suitable straight line, b_o can be determined from the slope of the line. We find the slope to be .0006 or $b_o \log e = .0060$ and $b_o = .0014$.

The remaining calculations can be set up as follows:

s	r	log r
400	.39	-.4089
750	.90	-.0458
1250	1.59	.2014
1750	3.32	.5211
2500	8.75	.9420

	$b_o s$	coth $b_o s$	w= s coth $b_o s$	$w-\bar{w}$	$(w-\bar{w})^2$
	.56	1.97	788	-684	$.468 \times 10^6$
	1.05	1.28	960	-512	$.262 \times 10^6$
	1.75	1.06	1325	-147	$.022 \times 10^6$
	2.45	1.02	1785	313	$.098 \times 10^6$
	3.50	1.00	2500	1028	1.056×10^6
Sum:			7358		1.906×10^6
Average:			$\bar{w}$ = 1472		

	log sinh $b_o s$	log r - log sinh $b_o s$	$(w-\bar{w})$ x prec.
	-.2293	-.1796	123
	-.0982	-.1440	74
	.4457	-.2443	36
	.7597	-.2386	- 75
	1.2186	-.2766	-285
Sum:		-1.0831	-127
Average		A_o = - .2166	

$$\Delta b = \frac{-127}{.4343 \times 1.906 \times 10^6} = -1.54 \times 10^{-4}$$

Let $\Delta b = -1.5 \times 10^{-4}$. Then

$$\Delta A = .4343\ (1.5 \times 10^{-4})(1472) = .0969$$

The corrected values of b and A are b = .00125

A = -.1206

Using log a = A, this gives a = .758. The final formula is

$$r = .758 \sinh .00125s$$

or

$$r = .758 \sinh \left(\frac{s}{800} \right).$$

The calculated values of r are compared with the observed values in the following table:

s	r obs.	r calc.	per cent error
400	.39	.393	+1.3
750	.90	.820	-8.9
1250	1.50	1.73	+8.8
1750	3.32	3.34	+ .6
2500	8.75	8.60	-1.7

The graph is indicated in fig 2.

Derivation of the equations.

To minimize (4) set its partial derivatives with respect to A and b equal to zero. The following equations are obtained:

(8) $$\sum (\log r_i - A - \log \sinh bs_i) = 0$$

(9) $$\sum s_i \coth bs_i \, (\log r_i - A - \log \sinh bs_i) = 0$$

From (8)

(10) $$A = \frac{1}{n} \sum (\log r_i - \log \sinh bs_i)$$

In (9) the "weights" $s_i \coth bs_i$ are approximated by $w_i = s_i \coth b_o s_i$.

Solving for A in (9)

$$\bar{w}A = \frac{1}{n} \sum w_i (\log r_i - \log \sinh bs_i)$$

Eliminating A between this equation and (10) the equation for b is as follows:

(11) $$\sum (w_i - \bar{w}) (\log r_i - \log \sinh bs_i) = 0$$

Letting $b = b_o + \Delta b$, then

(12) $$\log \sinh bs_i = \log \sinh b_o s_i + s_i \coth b_o s_i \cdot \Delta b (.4343)$$

approximately. Substituting (12) in (11) and solving for Δb,

(13) $$\Delta b = \frac{\sum (w_i - \bar{w})(\log r_i - \log \sinh b_o s_i)}{.4343 \cdot \sum (w_i - \bar{w})^2}$$

since $w_i = s_i \coth b_o s_i$ and also $\sum w_i(w_i - \bar{w}) = \sum (w_i - \bar{w})^2$.

Let the first approximation to A be

$$A_o = \frac{1}{n} \sum (\log r_i - \log \sinh b_o s_i) ,$$

Substitute (12) in (10) and let $A = A_o + \Delta A$, we find

$$\Delta A = - \frac{1}{n} \sum w_i \Delta b(.4343)$$

or

(14) $$\Delta A = -.4343\, \bar{w} \cdot \Delta b.$$

Summary

This paper demonstrates how to fit a hyperbolic sine curve to a set of experimental data. A least squares solution leads to a pair of transcendental equations. These are solved by iteration. An example is worked out which shows how closely this law fits the data taken from the Creep Rate Book.

References:

(1) "Ein Gedankenmodell zur kinetischen Theorie der festen Körper", by L. Prandtl, Zeit. für Ang. Math. und Mech. Vol. 8, 1928, pp. 85-106.

(2) "The Influence of Time upon Creep. The Hyperbolic Sine Creep Law", by A. Nadai. Stephen Timoshenko Anniversary Volume. Macmillan Co., 1938.

(3) "Viscosity, Plasticity and Diffusion as Examples of Absolute Reaction Rates", by H. Eyring. Jour. Chem. Physics, Vol. 4, 1936, pp. 283-291.

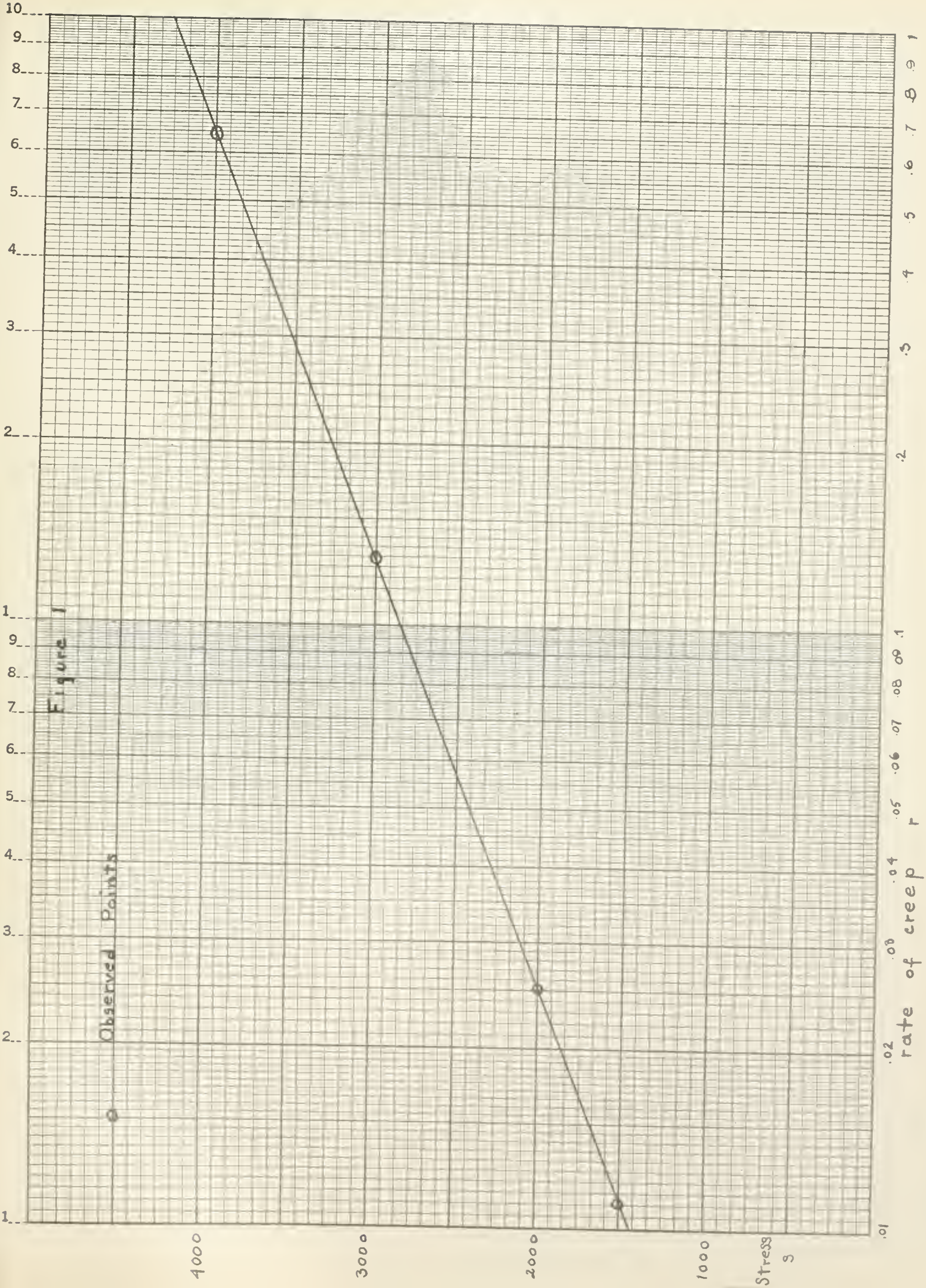

Figure 1

Observed Points
rate of creep r
Stress s
4000
3000
2000
1000
.01
.02
.03
.04
.05
.06
.07
.08
.09
.1
.2
.3
.4
.5
.6
.7
.8
.9
1
10
9
8
7
6
5
4
3
2
1

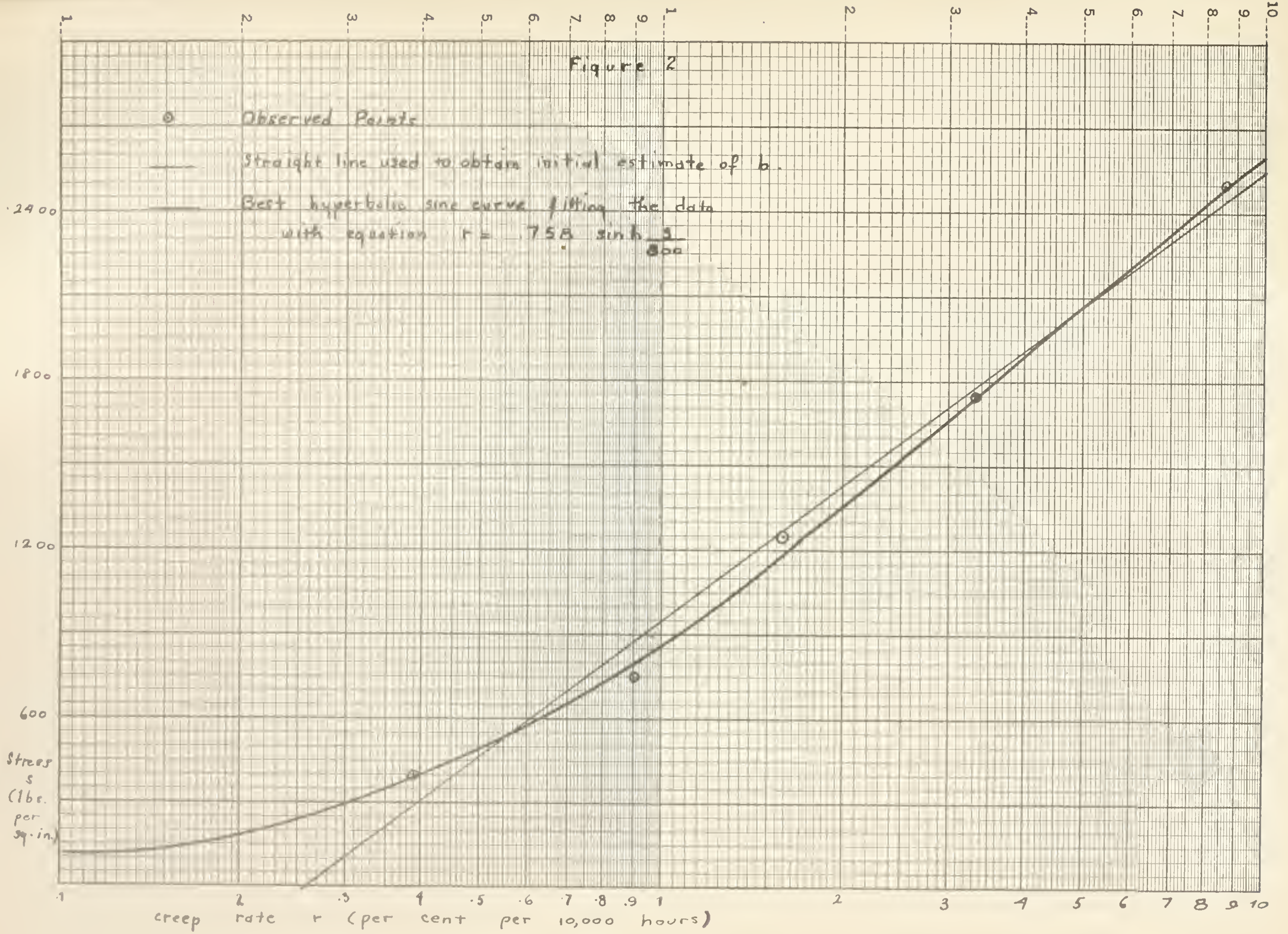

Figure 2

Observed Points
Straight line used to obtain initial estimate of b.
Best hyperbolic sine curve fitting the data with equation $r = .758 \sinh \frac{s}{800}$
2400
1800
1200
600
Stress s (lbs. per sq. in.)
.1
.2
.3
.4
.5
.6
.7
.8
.9
1
2
3
4
5
6
7
8
9
10
creep rate r (per cent per 10,000 hours)

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